

SOME PROPERTIES OF NEGATIVE PINCHED RIEMANNIAN MANIFOLDS OF DIMENSIONS 5 AND 7

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1. Let M be a compact orientable Riemannian manifold, and denote by $K^p(M, \mathbf{R})$ the vector space of Killing p -forms of the manifold M over the field \mathbf{R} of real numbers. It has been shown [3] that if the manifold M is negative k -pinched and of even dimension $n = 2m$ (resp. odd dimension $n = 2m + 1$), and $k > 1/4$ (resp. $k > 2(m - 1)/(8m - 5)$), then $K^2(M, \mathbf{R}) = 0$. In this paper, we have improved the above result for negative pinched manifolds of dimensions 5 and 7.

2. We consider a compact orientable negative k -pinched Riemannian manifold M . If α, β are two exterior p -forms of the manifold, then the local product of the two forms α, β and the norm of α are defined by

$$(\alpha, \beta) = \frac{1}{p!} \alpha^{i_1 \dots i_p} \beta_{i_1 \dots i_p} = \frac{1}{p!} \alpha_{i_1 \dots i_p} \beta^{i_1 \dots i_p},$$

$$|\alpha|^2 = \frac{1}{p!} \alpha^{i_1 \dots i_p} \alpha_{i_1 \dots i_p}.$$

If η is the volume element of the manifold M , then the global product of the two exterior p -forms α, β and the global norm of α are defined by

$$\langle \alpha, \beta \rangle = \int_M (\alpha, \beta) \eta,$$

$$\|\alpha\|^2 = \int_M |\alpha|^2 \eta.$$

It is well known that the following relation holds [1, p. 187]:

$$(2.1) \quad \langle \alpha, \Delta \alpha \rangle = \|\delta \alpha\|^2 + \|d\alpha\|^2.$$

We also have the formula [2, p. 3]:

$$(2.2) \quad \frac{1}{2} \Delta(|\alpha|^2) = (\alpha, \Delta \alpha) - |\nabla \alpha|^2 + \frac{1}{2(p-1)!} Q_p(\alpha),$$

where

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$$(2.3) \quad Q_p(\alpha) = (p-1)R_{klmn}\alpha^{klis\dots ip}\alpha^{mn}_{i_3\dots i_p} - 2R_{kl}\alpha^{ki_2\dots ip}\alpha^l_{i_2\dots i_p},$$

$$(2.4) \quad |\nabla\alpha|^2 = \frac{1}{p!}\nabla^k\alpha^{i_1\dots i_p}\nabla_k\alpha_{i_1\dots i_p}.$$

If $\alpha \in K^p(M, R)$, then it is easy to prove, using the property of α [4, p. 66]:

$$\nabla_X\alpha(Y, X_2, \dots, X_p) + \nabla_Y\alpha(X, X_2, \dots, X_p) = 0, \\ \text{for } Y, X, X_l \in T(M),$$

and the relation

$$(2.5) \quad (\alpha, \Delta\alpha) = -(p+1)Q_p(\alpha)/p!,$$

where $l = 2, \dots, p$.

Let P be a point of the manifold M , and consider a normal coordinate system on the manifold with origin at the point P . It is well known that there is an orthonormal basis $\{X_1, \dots, X_n\}$ in the tangent space M_p such that its dual basis $\{X_1^*, \dots, X_n^*\}$ has the property that the exterior 2-form α at the point P takes the form

$$(2.6) \quad \alpha = \alpha_{12}X_1^* \wedge X_2^* + \alpha_{34}X_3^* \wedge X_4^* + \dots + \alpha_{2m-1, 2m}X_{2m-1}^* \wedge X_{2m}^*.$$

where $m = [n/2]$.

Since the manifold M is negative k -pinched, the components of the Riemannian curvature tensor at the point P satisfy the relations and the inequalities [3]:

$$\langle R(X_i, X_j)X_l, X_h \rangle = R_{ijhl}, \\ \sigma_{ij} = \sigma(X_i, X_j) = R_{ijij}, \\ (2.7) \quad -1 \leq \sigma_{ij} \leq -k, \quad |R_{ijii}| \leq \frac{1}{2}(1-k), \quad |R_{ijhl}| \leq \frac{2}{3}(1-k),$$

where $i \neq j \neq h \neq l$.

3. Suppose that the manifold M is of dimension 5, and let α be an element of the vector space $K^2(M, R)$. Then we form the following exterior 4-form

$$(3.1) \quad \beta = \frac{1}{2}\alpha \wedge \alpha.$$

In this case, the formula (2.6) takes the form

$$(3.2) \quad \alpha = \alpha_{12}X_1^* \wedge X_2^* + \alpha_{34}X_3^* \wedge X_4^*.$$

The relation (3.1) by virtue of (3.2) becomes

$$(3.3) \quad \beta = \alpha_{12}\alpha_{34}X_1^* \wedge X_2^* \wedge X_3^* \wedge X_4^* .$$

From (3.2) and (3.3) we obtain

$$(3.4) \quad |\alpha|^2 = \alpha_{12}^2 + \alpha_{34}^2, \quad |\beta| = \alpha_{12}\alpha_{34} .$$

For the exterior 4-form β the formula (2.4) becomes

$$(3.5) \quad |\nabla\beta|^2 = \nabla^k\beta^{i_1i_2i_3i_4}\nabla_k\beta_{i_1i_2i_3i_4}, \quad i_1 < i_2 < i_3 < i_4 .$$

In the general case, the coefficients $\beta_{i_1i_2i_3i_4}$ of the exterior 4-form β are given by

$$(3.6) \quad \beta_{i_1i_2i_3i_4} = \alpha_{i_1i_2}\alpha_{i_3i_4} + \alpha_{i_1i_3}\alpha_{i_4i_2} + \alpha_{i_1i_4}\alpha_{i_2i_3} .$$

By means of (3.6) and from the fact that α is a Killing 2-form, the relation (3.5) becomes

$$(3.7) \quad |\nabla\beta|^2 \leq \alpha_{12}^2T_1 + \alpha_{34}^2T_2 ,$$

where T_1 and T_2 are linear expressions of terms of the form $(\nabla_{\lambda}^{\mu}\alpha_{\nu\rho})^2$ whose coefficients are 0, 1, 4. Since α is a Killing 2-form, we have

$$(3.8) \quad |\nabla\alpha|^2 = 3(\nabla_{\lambda}^{\mu}\alpha_{\nu\rho})^2 .$$

From (3.7) and (3.8) and the property of T_1, T_2 we obtain the inequality

$$(3.9) \quad |\nabla\beta|^2 \leq \frac{4}{3}|\nabla\alpha|^2|\alpha|^2 .$$

If we estimate $\frac{1}{2}Q_2(\alpha)$ from the formula (2.3), we have

$$\begin{aligned} \frac{1}{2}Q_2(\alpha) = & -(\sigma_{13} + \sigma_{14} + \sigma_{15} + \sigma_{23} + \sigma_{24} + \sigma_{25})\alpha_{12}^2 \\ & -(\sigma_{31} + \sigma_{32} + \sigma_{35} + \sigma_{41} + \sigma_{42} + \sigma_{45})\alpha_{34}^2 \\ & + 4R_{1234}\alpha_{12}\alpha_{34} , \end{aligned}$$

which gives the inequality, by means of (2.7) and (3.4),

$$(3.10) \quad \frac{1}{2}Q_2(\alpha) \geq 6k|\alpha|^2 - \frac{8}{3}(1-k)|\beta| .$$

If we also estimate $\frac{1}{2}Q_4(\beta)$ from the same formula (2.3), we obtain

$$\frac{1}{2}Q_4(\beta) = -3!(\sigma_{15} + \sigma_{25} + \sigma_{35} + \sigma_{45})\alpha_{12}^2\alpha_{34}^2 ,$$

which implies the inequality, by means of the first of (2.7) and the second of (3.4),

$$(3.11) \quad \frac{1}{2}Q_4(\beta) \geq 4!k|\beta|^2.$$

It is clear that the above calculations have been done at the point P with respect to the special orthonormal frame in the tangent space M_P .

4. If we integrate the formula (3.9), we obtain

$$(4.1) \quad \|\nabla\beta\|^2 \leq \frac{4}{3} \int_M |\alpha|^2 |\nabla\alpha|^2 \eta.$$

The relation (2.2) for the exterior 4-form β becomes

$$\frac{1}{2}d(|\beta|^2) = (\beta, \Delta\beta) - \|\nabla\beta\|^2 + \frac{1}{6 \cdot 2}Q_4(\beta),$$

from which we have

$$(4.2) \quad 0 = \int_M (\beta, \Delta\beta)\eta - \|\nabla\beta\|^2 + \frac{1}{6} \int_M \frac{1}{2}Q_4(\beta)\eta.$$

By means of (2.1) and (3.11), the above equation (4.2) gives

$$\|d\beta\|^2 + \|\delta\beta\|^2 - \|\nabla\beta\|^2 + 4k\|\beta\|^2 \leq 0,$$

or finally

$$(4.3) \quad \|\nabla\beta\|^2 \geq 4k\|\beta\|^2.$$

It is well known that the following formula holds

$$\frac{1}{2}d(|\alpha|^4) = |\alpha|^2 d(|\alpha|^2) - (d(|\alpha|^2))^2,$$

from which we obtain

$$(4.4) \quad \int_M |\alpha|^2 d(|\alpha|^2)\eta = \int_M (d(|\alpha|^2))^2 \eta \geq 0.$$

Since α is a Killing 2-form, (2.2) takes the form, by means of (2.5),

$$\frac{1}{2}d(|\alpha|^2) = -|\nabla\alpha|^2 - \frac{1}{4}Q_2(\alpha),$$

which, by integration of the manifold M and the inequalities (3.10) and (4.4), gives the inequality

$$(4.5) \quad 3 \int_M |\alpha|^2 |\nabla \alpha|^2 \eta \leq \int_M [4(1 - k) |\beta| |\alpha|^2 - 9k |\alpha|^4] \eta.$$

The inequality (4.5) together with (4.1) and (4.3) implies

$$9 \|\beta\|^2 k \leq \int_M [4(1 - k) |\beta| |\alpha|^2 - 9k |\alpha|^4] \eta,$$

or

$$\int_M [9k |\alpha|^4 - 4(1 - k) |\beta| |\alpha|^2 + 9 |\beta|^2 k] \eta \leq 0.$$

Let f be the function defined by

$$f = 9k |\alpha|^4 - 4(1 - k) |\beta| |\alpha|^2 + 9 |\beta|^2 k,$$

which at the point P takes the form

$$(4.6) \quad f = 9k(\alpha_{12}^2 + \alpha_{34}^2)^2 - 4(1 - k)\alpha_{12}\alpha_{34}(\alpha_{12}^2 + \alpha_{34}^2) + 9k\alpha_{12}^2\alpha_{34}^2.$$

It is easy to show that if $k > 8/53$, then $f \geq 0$, where the equality holds if $\alpha_{12} = \alpha_{34} = 0$.

From the above we derive

Theorem 1. *Let M be a compact orientable negative k -pinched manifold of dimension 5. If $k > 8/53$, then $K^2(M, \mathbf{R}) = 0$.*

5. We assume that the manifold M is of dimension 7. In this case, the relation (2.6) becomes

$$(5.1) \quad \alpha = \alpha_{12}X_1^* \wedge X_1^* + \alpha_{34}X_3^* \wedge X_4^* + \alpha_{56}X_5^* \wedge X_6^*.$$

Let γ be the exterior 6-form defined by

$$\gamma = \frac{1}{3!} \alpha \wedge \alpha \wedge \alpha,$$

which by means of (5.1) becomes

$$(5.2) \quad \gamma = \alpha_{12}\alpha_{34}\alpha_{56}X_1^* \wedge \dots \wedge X_6^*.$$

From (5.1) and (5.2) we obtain

$$(5.3) \quad |\alpha|^2 = \alpha_{12}^2 + \alpha_{34}^2 + \alpha_{56}^2, \quad |\gamma| = \alpha_{12}\alpha_{34}\alpha_{56}.$$

If we apply the same technique as in § 3, we obtain, in this case, the inequalities

$$(5.4) \quad \frac{1}{2} Q_2(\alpha) \geq 10k |\alpha|^2 - \frac{8}{3} (1 - k)\theta,$$

$$(5.5) \quad \frac{1}{2} Q_6(\gamma) \geq 6! k |\gamma|^2,$$

where

$$(5.6) \quad \theta = \alpha_{12}\alpha_{34} + \alpha_{34}\alpha_{56} + \alpha_{56}\alpha_{12}.$$

In the general case, the coefficients $\gamma_{\nu_1 \dots \nu_6}$ of the exterior 6-form γ are given by

$$(5.7) \quad \gamma_{\nu_1 \dots \nu_6} = \alpha_{\nu_1 \nu_2} A + \alpha_{\nu_1 \nu_3} B + \alpha_{\nu_1 \nu_4} C + \alpha_{\nu_1 \nu_5} D + \alpha_{\nu_1 \nu_6} E,$$

where

$$A = \alpha_{\nu_3 \nu_4} \alpha_{\nu_5 \nu_6} + \alpha_{\nu_3 \nu_5} \alpha_{\nu_6 \nu_4} + \alpha_{\nu_3 \nu_6} \alpha_{\nu_4 \nu_5},$$

$$B = \alpha_{\nu_2 \nu_4} \alpha_{\nu_6 \nu_5} + \alpha_{\nu_2 \nu_5} \alpha_{\nu_4 \nu_6} + \alpha_{\nu_2 \nu_6} \alpha_{\nu_5 \nu_4},$$

$$C = \alpha_{\nu_2 \nu_3} \alpha_{\nu_5 \nu_6} + \alpha_{\nu_2 \nu_5} \alpha_{\nu_6 \nu_3} + \alpha_{\nu_2 \nu_6} \alpha_{\nu_3 \nu_5},$$

$$D = \alpha_{\nu_2 \nu_3} \alpha_{\nu_6 \nu_4} + \alpha_{\nu_2 \nu_4} \alpha_{\nu_3 \nu_6} + \alpha_{\nu_2 \nu_6} \alpha_{\nu_4 \nu_3},$$

$$E = \alpha_{\nu_2 \nu_3} \alpha_{\nu_4 \nu_5} + \alpha_{\nu_2 \nu_4} \alpha_{\nu_5 \nu_3} + \alpha_{\nu_2 \nu_5} \alpha_{\nu_3 \nu_4}.$$

The formula (2.4) for the exterior 6-form γ becomes

$$|\nabla \gamma|^2 = \nabla^k \gamma^{i_1 \dots i_6} \nabla_k \gamma_{i_1 \dots i_6}, \quad i_1 < i_2 < \dots < i_6,$$

which, by means of (5.7) and from the fact that α is a Killing 2-form, is reduced to

$$(5.8) \quad |\nabla \gamma|^2 \leq \alpha_{12}^2 \alpha_{34}^2 \sum_1 + \alpha_{34}^2 \alpha_{56}^2 \sum_2 + \alpha_{56}^2 \alpha_{12}^2 \sum_3,$$

where \sum_1, \sum_2, \sum_3 are linear expressions of the terms of the form $(\nabla_\lambda \alpha_{\mu\nu})^2$ whose coefficients are 0, 1, 2, 5.

From (3.8), (5.8) and the property of \sum_1, \sum_2, \sum_3 we derive the inequality

$$|\nabla \gamma|^2 \leq \frac{5}{3} |\nabla \alpha|^2 (\alpha_{12}^2 \alpha_{34}^2 + \alpha_{34}^2 \alpha_{56}^2 + \alpha_{56}^2 \alpha_{12}^2),$$

which, by means of the inequality

$$(\alpha_{12}^2 + \alpha_{34}^2 + \alpha_{56}^2)^2 \geq 3(\alpha_{12}^2 \alpha_{34}^2 + \alpha_{34}^2 \alpha_{56}^2 + \alpha_{56}^2 \alpha_{12}^2),$$

takes the form

$$(5.9) \quad |\nabla\gamma|^2 \leq \frac{5}{9} |\nabla\alpha|^2 |\alpha|^4.$$

6. From (5.9) we obtain

$$(6.1) \quad \frac{9}{5} \|\nabla\gamma\|^2 \leq \int_M |\alpha|^4 |\nabla\alpha|^2 \eta.$$

It is well known that the following relation holds

$$\frac{1}{3} \Delta(|\alpha|^6) = |\alpha|^4 \Delta(|\alpha|^2) - 2|\alpha|^2 (d(|\alpha|^2))^2,$$

which implies

$$(6.2) \quad \int_M |\alpha|^4 \Delta(|\alpha|^2) \eta \geq 0.$$

Since α is a Killing 2-form, then the relation (2.2), by virtue of (2.5), becomes

$$\frac{1}{2} \Delta(|\alpha|^2) = -|\Delta\alpha|^2 - \frac{1}{4} Q_2(\alpha),$$

or

$$\frac{1}{2} |\alpha|^4 \Delta(|\alpha|^2) = -|\alpha|^4 |\nabla\alpha|^2 - |\alpha|^4 \frac{1}{4} Q_2(\alpha),$$

from which by integration on the manifold M and by the inequalities (5.4) and (6.2) we obtain

$$(6.3) \quad 3 \int_M |\alpha|^4 |\nabla\alpha|^2 \eta \leq \int_M [4(1-k)\theta |\alpha|^4 - 15k |\alpha|^6] \eta.$$

The formula (2.2) for the 6-form γ becomes

$$\frac{1}{2} \Delta(|\gamma|^2) = (\gamma, \Delta\gamma) - |\nabla\gamma|^2 + \frac{1}{5! \cdot 2} Q_6(\gamma),$$

from which by integration on the manifold M we have

$$0 = \int_M (\gamma, \Delta\gamma) \eta - \|\nabla\gamma\|^2 + \frac{1}{5!} \int_M \frac{1}{2} Q_6(\gamma) \eta,$$

which, by means of (2.1) and (5.5), takes the form

$$\|d\gamma\|^2 + \|\delta\gamma\|^2 - \|\nabla\gamma\|^2 + 6k\|\gamma\|^2 \leq 0,$$

or

$$(6.4) \quad \|\nabla\gamma\|^2 \geq 6k\|\gamma\|^2.$$

From the inequalities (6.1), (6.3) and (6.4) we derive the inequality

$$\int_M [75|\alpha|^6 k - 20|\alpha|^4 \theta(1-k) + 162k|\gamma|^2] \eta \leq 0.$$

We denote by F the following function

$$F = 75|\alpha|^6 k - 20|\alpha|^4 \theta(1-k) + 162k|\gamma|^2,$$

which, by means of (5.3) and (5.6), takes the form, at the point P ,

$$(6.5) \quad F = 75k(\alpha_{12}^2 + \alpha_{34}^2 + \alpha_{56}^2)^3 - 20(1-k)(\alpha_{12}^2 + \alpha_{34}^2 + \alpha_{56}^2)^2 \\ \cdot (\alpha_{12}\alpha_{34} + \alpha_{34}\alpha_{56} + \alpha_{56}\alpha_{12}) + 162k(\alpha_{12}\alpha_{34}\alpha_{56})^2.$$

It is easy to show the inequalities

$$\alpha_{12}\alpha_{34} + \alpha_{34}\alpha_{56} + \alpha_{56}\alpha_{12} \leq \alpha_{12}^2 + \alpha_{34}^2 + \alpha_{56}^2, \\ (6.6) \quad 27(\alpha_{12}\alpha_{34}\alpha_{56})^2 \leq (\alpha_{12}^2 + \alpha_{34}^2 + \alpha_{56}^2)^3.$$

From (6.5) and the inequalities (6.6) we conclude that if $k > 20/101$, then $F \geq 0$, where the equality holds if $\alpha_{12} = \alpha_{34} = \alpha_{56} = 0$.

From the above we derive

Theorem II. *Let M be a compact orientable negative k -pinched manifold of dimension 7. If $k > 20/101$, then $K^2(M, \mathbb{R}) = 0$.*

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